Math 432: Set Theory and Topology HOMEWORK 6 **Due: March 14/15**

1. Let $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) := n^2$ and $g(n) := n^3$, for $n \in \mathbb{N}$. Explicitly define a bijection $h : \mathbb{N} \to \mathbb{N}$ such that $h \subseteq f \cup g^{-1}$, where by g^{-1} we mean the graph of g with the swapped coordinates, i.e., $g^{-1} := \{(y, x) \in \mathbb{N}^2 : g(x) = y\}$.

HINT: Run the proof of the Cantor–Schröder–Bernstein theorem.

2. For a set A containing and an element $a_0 \in A$, let $A^{\mathbb{N}}_{a_0}$ denote the set of all sequences of elements of A that are eventually a_0, i.e.,

$$A^{\mathbb{N}}_{a_0} := \left\{x \in A^{\mathbb{N}} : \exists N \in \mathbb{N}\, \forall n \geqslant N\; x(n) = a_0\right\}.$$

(a) Prove that $A^{\mathbb{N}}_{a_0} \sqsubseteq A^{<\omega}$.

(b) (Optional[1]) Prove that if A has at least two distinct elements, then $A^{\mathbb{N}}_{a_0} \equiv A^{<\omega}$.

3. Let A, B, C be sets.

(a) Prove: if $A \equiv \mathbb{N}$ and B is countable, then $A \cup B \equiv A$.

(b) Conclude that if $\mathbb{N} \sqsubseteq C$ and B is countable, then $\mathbb{N} \cup B \sqsubseteq C$.

4. Let A be a set.

(a) Prove: if $\mathbb{N} \sqsubseteq A$, then A is Dedekind infinite. (A picture of the proof was drawn in class.)

(b) Carefully prove the converse (this was also proven in class): if A is Dedekind infinite, then $\mathbb{N} \sqsubseteq A$. Did you use Axiom of Choice?

5. For a partial ordering $(A, <)$, a sequence $x \in A^{\mathbb{N}}$ is said to be decreasing if for all $n \in \mathbb{N}$, $x(n) > x(n+1)$. Prove that $(A, <)$ is a well-ordering if and only if it is a total ordering and there are no decreasing sequences in it. Did you use Axiom of Choice?

6. Let A, B be sets.

(a) Recall that in general, to get an injection $g : B \hookrightarrow A$ from a surjection $f : A \twoheadrightarrow B$, one has to use Axiom of Choice. Why and what is the construction?

(b) In case A is well-ordered (i.e., there is a well-order $<$ on A), do we have to use Axiom of Choice to get an injection $g : B \hookrightarrow A$ from a surjection $f : A \twoheadrightarrow B$?

(c) Do we need Axiom of Choice in the implication $A \twoheadrightarrow \mathbb{N} \implies \mathbb{N} \sqsubseteq A$?

[1] It is good to prove this, but you will not get credit for it.